Fisher-Hartwig conjecture and the correlators in XY spin chain.

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Abstract

We apply the theorems from the theory of Toeplitz determinants to calculate the asymptotics of the correlators in the XY spin chain in the transverse magnetic field. The asymptotics of the correlators for the XX spin chain in the magnetic field are obtained.

1. Introduction

The calculation of correlators in the one-dimensional exactly solvable models remains an interesting and important problem. An important exact results on the correlators for the XY quantum spin chain and the Ising model were obtained [1]-[3]. The goal of the present letter is twofold. First, we present the calculations of the correlators in the limit of the XX spin chain in a non-zero magnetic field which was not considered previously. In particular, the asymptotics of the exponential correlator for the XX spin chain is found. Second, we use the generalized Fisher-Hartwig conjecture (for example, see [4]) and the method to calculate the asymptotics of Toeplitz determinants [5] to derive the known results [3] for the correlators in the XY spin chain in a non-zero magnetic field in a simple way.

In the present letter to calculate the asymptotics of the determinants we use the conjecture from the theory of Toeplitz matrices, [4], the generalized Fisher-Hartwig conjecture, which is based on the proofs of the original Fisher-Hartwig conjecture [6] including the constant in front of the asymptotics, in a number of the particular cases [7], [8] (see also [4] and references therein). One finds the representation of the generating function $f(x)$ of the Toeplitz matrix $M_{ij} = M(i-j) = \int_0^{2\pi} (dx/2\pi) e^{i(i-j)x} f(x)$ of the following form:

$$f(x) = f_0(x) \prod_r e^{ib_r(x - x_r - \pi \mathrm{sign}(x - x_r))} (2 - 2\cos(x - x_r))^{a_r}, \qquad (1)$$

where $x \in (0, 2\pi)$ is implied, the discontinuities (jumps and zeroes or the power-law singularities) are at finite number of the points x_r, and $f_0(x)$ is the smooth non-vanishing function with the continuously defined argument at the interval $(0, 2\pi)$. The function (1) is characterized by the parameters a_r, b_r at each point of discontinuity x_r. In general there are several representations of the form (1) for a given functions $f(x)$. To obtain the asymptotics of the determinant one should take the sum over the representations (1) corresponding to the minimal exponent $\sum_r (b_r^2 - a_r^2)$:

$$D(N) = \sum_{Repr.} e^{l_0 N} N^{\sum_r (a_r^2 - b_r^2)} E \qquad (2)$$

where E is the constant independent of N,

$$E = \exp\left(\sum_{k=1}^{\infty} k l_k l_{-k}\right) \prod_r (f_+(x_r))^{-a_r+b_r} (f_-(x_r))^{-a_r-b_r} \tag{3}$$

$$\prod_{r\neq s}\left(1-e^{i(x_s-x_r)}\right)^{-(a_r+b_r)(a_s-b_s)} \prod_r \frac{G(1+a_r+b_r)G(1+a_r-b_r)}{G(1+2a_r)},$$

$l_k = \int_{-\pi}^{\pi}(dx/2\pi)e^{ikx}\ln(f_0(x))$, and the functions $f_\pm(x)$ are given by the equations

$$\ln f_+(x) = \sum_{k>0} l_{-k}e^{ikx}, \quad \ln f_-(x) = \sum_{k>0} l_k e^{-ikx},$$

where G is the Barnes G-function [9], $G(z+1) = G(z)\Gamma(z)$, $G(1) = 1$.

For the piecewise continuous generation function $f(x)$ with the finite number of discontinuities which is not equal to zero ($a_r = 0$) the equation (2) can be represented in a more convenient form [7]. Suppose the function $f(x)$ has the finite number of discontinuities at the points x_r,

$$\lambda_r = \frac{1}{2\pi}\left(\ln f(x_r+0) - \ln f(x_r-0)\right),$$

and assume that the function $f(x)$ has the continuously defined argument and not equal to zero at the interval $(-\pi,\pi)$. Then the asymptotic of the determinant is given by the formula [7]

$$D(N) = \sum_{Repr.} e^{l_0 N} N^{(\sum_r \lambda_r^2)} E, \quad E = \exp\left(\sum_{k=1}^{\infty}\left(k l_k l_{-k} - \frac{1}{k}\sum_r \lambda_r^2\right)\right)\prod_r \tilde{g}(\lambda_r), \tag{4}$$

where we denote $l_k = \int_{-\pi}^{\pi}(dx/2\pi)e^{ikx}\ln(f(x))$ and the function $\tilde{g}(\lambda)$ equals

$$\tilde{g}(\lambda) = e^{(1+\gamma)\lambda^2}\prod_{k=1}^{\infty}\left(1+\frac{\lambda^2}{k^2}\right)e^{-\lambda^2/k}, \tag{5}$$

where $\gamma = 0.577..$ is Euler's constant. The constant E (5) in eq.(4) is a consequence of the formula for the Barnes G- function:

$$G(1+z) = (2\pi)^{z/2}\exp\left(-\frac{z(z+1)}{2} - \gamma\frac{z^2}{2}\right)\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^k \exp(-z+z^2/2k),$$

in such a way that $\tilde{g}(\lambda_r) = G(1+z)G(1-z)$ for $\lambda_r = iz$.

In ref.[7] the theorem was proved for an arbitrary number of the discontinuities of the imaginary part of the magnitude less than 1/2, $|\lambda_r| < 1/2$, however, there are many reasons to believe it to be true also in the case $|\lambda_r| = 1/2$ [4]. For the case of an arbitrary single Fisher-Hartwig singularity the conjecture (2) was recently proved in ref.[8]. See ref.[4] for the complete list of the cases for which the rigorous proof of the conjecture (2) is available. Schematically the rigorous proofs in the particular cases [7] [8] go as follows. Suppose that equation (4) is fulfilled for some functions $f_1(x)$ and $f_2(x)$ of the class of piecewise continuous functions

with continuously defined argument. Then the equation (4) is fulfilled for the function $f(x) = f_1(x)f_2(x)$. Thus it is sufficient to prove eq.(4) for the smooth function with the continuously defined argument, in which case it is reduces to the well known strong Szego theorem and for the singular function of the form (1) $f(x) = (1-z)^\alpha(1-1/z)^\beta$, $z = e^{ix}$, in which case the asymptotics is known exactly.

2. General form of the correlators for XY spin chain.

Let us review the known calculations [1]-[3] of the equal-time correlators of the operators $S^\pm$ for the XY spin chain at non-zero transverse magnetic field. The Hamiltonian of the XY spin chain has the form

$$H = -\sum_{i=1}^{L}\left((1+\gamma)S_i^x S_{i+1}^x + (1-\gamma)S_i^y S_{i+1}^y - hS_i^z\right), \tag{6}$$

where $S^a = \frac{1}{2}\sigma^a$, $a = x, y, z$, $0 < \gamma < 1$ and the periodic boundary conditions are implied. Performing the well-known Jordan-Wigner transformation ($S^\pm = S^x \pm iS^y$):

$$S_x^+ = e^{i\pi N(x)}a_x^+ = \exp(\sum_{l<x} n_l)a_x^+, \tag{7}$$

we obtain the following Hamiltonian written down in terms of the fermionic operators:

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left(a_i^+ a_{i+1} + \gamma a_i^+ a_{i+1}^+ + h.c.\right) + h\sum_{i=1}^{L} a_i^+ a_i, \tag{8}$$

where the boundary terms which are not important in the thermodynamic limit are omitted. The Hamiltonian is diagonalized with the help of the following Fourier transform in the sectors with an odd number of particles:

$$a_k^+ = \frac{1}{\sqrt{L}}\sum_x e^{ikx}a_x^+, \quad k = \frac{2\pi n}{L}, \quad n \in Z, \quad -\pi < k < \pi.$$

The Hamiltonian (8) is easily diagonalized in the momentum space. Introducing the column $\psi_k^+ = (a_k^+, a_{-k})$ for $k \in (0, \pi)$ the Hamiltonian is represented as

$$H = -\sum_{k>0}\psi_k^+ \hat{M}_k \psi_k, \quad \hat{M}_k = \begin{pmatrix} \epsilon_k & i\Delta_k \\ -i\Delta_k & -\epsilon_k \end{pmatrix}, \quad \psi_k = \begin{pmatrix} a_k \\ a_{-k}^+ \end{pmatrix}, \quad k > 0,$$

where the notations $\epsilon_k = \cos(k) - h$, $\Delta_k = \gamma\sin(k)$ are used and the additional constant term ϵ_k is implied. It is convenient to redefine the operator $a_{-k}^+ \to ia_{-k}^+$. In this case the Hamiltonian is diagonalized by means of the canonical transformation

$$a_k^+ = c_k\alpha_{1k}^+ + s_k\alpha_{2k}, \quad a_{-k} = c_k\alpha_{2k} - s_k\alpha_{1k}^+, \tag{9}$$

where $k > 0$, $c_k^2 + s_k^2 = 1$, $c_k = \cos(\phi_k)$, $s_k = \sin(\phi_k)$, and the angle ϕ_k and the Hamiltonian are:

$$\mathrm{tg}(2\phi_k) = -\frac{\Delta_k}{\epsilon_k}, \quad H = -\sum_{k>0}\left(\alpha^+_{1k}\alpha_{1k} - \alpha^+_{2k}\alpha_{2k} + 1\right), \quad E_k = \sqrt{(\cos(k)-h)^2 + \gamma^2\sin^2(k)}.$$

Consider the equal-time correlators for the XY spin chain:

$$G(x) = \langle 0|S^+_{i+x}S^-_i|0\rangle. \qquad \tilde{G}(x) = \langle 0|S^+_{i+x}S^+_i|0\rangle.$$

Using the Jordan-Wigner transformation $S^+_x = \exp(i\pi\sum_{l<x} n_l)a^+_x$, the correlation functions $G(x)$, $\tilde{G}(x)$ can be represented as the following averages over the ground state of the Hamiltonian (8),

$$G(x) = \langle 0|a^+_x e^{i\pi N(x)}a_0|0\rangle, \qquad \tilde{G}(x) = -\langle 0|a^+_x e^{i\pi N(x)}a^+_0|0\rangle, \tag{10}$$

where $\hat{N}(x) = \sum_{i=1}^{x-1} n_i$. Introducing the operators, anticommuting at different sites,

$$A_i = a^+_i + a_i, \quad B_i = a^+_i - a_i, \quad A_iB_i = e^{i\pi n_i},$$

where $n_i = a^+_i a_i$ - is the fermion occupation number, with the following correlators with respect to the vacuum,

$$\langle B_iA_j\rangle = BA(i-j), \quad \langle A_iA_j\rangle = 0, \quad \langle B_iB_j\rangle = 0$$

one obtains the following expression for the bosonic correlator:

$$G(x) = \frac{1}{4}\langle 0|B_x\prod_{i=1}^{x-1}(A_iB_i)A_0|0\rangle + \frac{1}{4}\langle 0|B_0\prod_{i=1}^{x-1}(A_iB_i)A_0|0\rangle = \langle S^y_xS^y_0\rangle + \langle S^x_xS^x_0\rangle, \tag{11}$$

where the terms $\langle S^y_xS^y_0\rangle$, $\langle S^x_xS^x_0\rangle$ correspond respectively to the two terms at the left-hand side of this equation. At zero magnetic field each term in this expression is represented as a product of the two Toeplitz determinants [1]. The asymptotic behaviour of $G(x)$ is determined by the form of the average $BA(x)$. To calculate this function one should use the expressions for the averages of two operators a^+_i, a_i, which follow from the canonical transformation (9):

$$\langle a^+_x a_0\rangle = \int_0^\pi \frac{dk}{2\pi}\left(e^{ikx} + e^{-ikx}\right)\left(\frac{1}{2} + \frac{1}{2}\frac{\cos(k)-h}{\sqrt{(\cos(k)-h)^2+\gamma^2\sin^2(k)}}\right), \tag{12}$$

$$\langle a^+_x a^+_0\rangle = \int_0^\pi \frac{dk}{2\pi}\left(e^{ikx} - e^{-ikx}\right)\left(\frac{1}{2}\frac{i\gamma\sin(k)}{\sqrt{(\cos(k)-h)^2+\gamma^2\sin^2(k)}}\right)$$

where the sums have been replaced by the integrals in the continuum limit. From the expressions (12) the following expression can be obtained:

$$BA(x) = \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{ikx}\frac{\cos(k)-h+i\gamma\sin(k)}{\sqrt{(\cos(k)-h)^2+\gamma^2\sin^2(k)}},$$

which gives the following generation functions $f_1(x)$, $f_2(x)$ for the correlators $\langle S^x_x S^x_0\rangle$, $\langle S^y_x S^y_0\rangle$ respectively for the XY spin chain in the magnetic field:

$$f_{1,2}(x) = e^{\mp ix}\mathrm{sign}_\gamma(x) = e^{\mp ix}\left(\frac{\cos(k) - h + i\gamma\sin(k)}{\cos(k) - h - i\gamma\sin(k)}\right)^{1/2}. \tag{13}$$

Finally, let us obtain the determinant representation for the exponential correlator

$$G_\alpha(x) = \langle e^{i\alpha N(x)}\rangle, \tag{14}$$

where α is an arbitrary parameter and $N(x) = \sum_{i=1}^{x} n_i$, introducing the operators

$$A_i = e^{i\alpha/2}a_i^+ + a_i, \quad B_i = a_i^+ + e^{i\alpha/2}a_i, \quad A_iB_i = e^{i\alpha n_i},$$

for an arbitrary α we represent the correlator (14) as the average $\langle\prod_{i=1}^{x}(a_iB_i)\rangle$. Since for $\alpha \neq \pi$ the averages $\langle A_iA_j\rangle$ and $\langle B_iB_j\rangle$ are not equal to zero at $\gamma \neq 0$, we obtain the correlator (14) as a Pfaffian, which can be expressed through the Toeplitz determinants. For the case $\alpha = \pi$ we have $\langle A_iA_j\rangle = \langle B_iB_j\rangle = 0$ and the average can be represented as the Toeplitz determinant $\det_{ij}(\langle A_iB_j\rangle)$, $i, j = 1, \ldots x$, At $\gamma = 0$ the generating function for this determinant equals $f(x) = \mathrm{sign}(|x| - p_F)$.

For the practical calculations one should represent the functions (13) in the following form:

$$f_{1,2}(x) = \left(e^{i2x}\right)\left(\frac{1 - \lambda_1 e^{-ix}}{1 - \lambda_1 e^{ix}}\right)^{1/2}\left(\frac{1 - \lambda_2 e^{-ix}}{1 - \lambda_2 e^{ix}}\right)^{1/2}, \tag{15}$$

where the factor e^{i2x} in the parenthesis corresponds to the correlator $\langle S^y_x S^y_0\rangle$ (to the function $f_2(x)$) and the parameters $\lambda_{1,2}$ are given by the expression:

$$\lambda_{1,2} = \frac{1}{1+\gamma}\left(h \mp \sqrt{h^2 + \gamma^2 - 1}\right). \tag{16}$$

The behaviour of the correlators is determined by the values of the parameters $\lambda_{1,2}$, which obey the relation $\lambda_1\lambda_2 = a$, where the parameter $a < 1$ equals $a = (1-\gamma)/(1+\gamma)$. In general, for $\gamma > 0$ one should distinguish four different cases. 1) $h^2 + \gamma^2 < 1$. In this case the parameters $\lambda_{1,2} = \sqrt{a}e^{\pm i\phi}$ are the complex numbers. 2) $h^2 + \gamma^2 > 1$ but $h < 1$. In this case the parameters $\lambda_{1,2}$ are real and obey $\lambda_1 < \lambda_2 < 1$. 3) The line $h = 1$ at arbitrary γ: $\lambda_1 = a < 1$, $\lambda_2 = 1$. 4) The region $h > 1$ where the real parameters $\lambda_{1,2}$ are $\lambda_1 < 1 < \lambda_2$. The asymptotic behaviour of the correlators is different in this four regions.

3. Correlators for the XX spin chain.

In this section we calculate the correlators for the isotropic XX spin chain ($\gamma = 0$) at non-zero magnetic field. For this purpose it is sufficient to use the equation (4). Let us present the generating functions $f(x)$ for the initial determinants of the $x \times x$ matrices obtained for different correlators. We construct the functions with the continuously defined argument at

the interval $(-\pi, \pi)$ which is important for the application of the equation (4) (see [7]). One can use the following functions. First for the correlator (14) at $\alpha < \pi$ one should use the function

$$f(x) = (1, e^{i\alpha}, 1) \tag{17}$$

where the three number in parenthesis denote the values of the function at the intervals $(-\pi, -p_F)$, $(-p_F, p_F)$ and (p_F, π) respectively and the unity in eq.(17) denotes the phase equal to zero, the function

$$f(x) = (1, -1, 1) \tag{18}$$

for the same correlator at $\alpha = \pi$ and the functions of the form

$$f_1(x) = e^{-ix}(e^{-i\pi}, 1, e^{i\pi}), \qquad f_2(x) = e^{ix}(e^{i\pi}, 1, e^{-i\pi}), \tag{19}$$

with the continuously defined argument for the two terms in eq.(11) respectively. Let us note that for the case $\alpha > \pi$ in eq.(14) the function can be chosen in the form $(e^{i2\pi}, e^{i\alpha}, e^{i2\pi})$, which shows that the correlator (14) is in fact the 2π - periodic function of the real parameter α.

The function $f(x)$ (17) has the continuously defined argument and the two jumps of the size $i\alpha$ and $-i\alpha$ at the points $x_1 = -\pi/2$ and $x_2 = \pi/2$. Substituting this functions into the equation (4) we obtain at the half-filling the asymptotics:

$$G_\alpha(x) = \langle e^{i\alpha N(x)} \rangle = e^{i\alpha x/2} \frac{1}{x^{\alpha^2/2\pi^2}} \frac{1}{2^{\alpha^2/2\pi^2}} \left(g(\lambda)\right)^2, \tag{20}$$

where $\lambda = \alpha/2\pi$. Evaluating the product in $g(\lambda)$ for the function given by the equation (5) we obtain the expression:

$$\ln(g(\lambda)) = \lambda^2 \int_0^\infty \frac{dt}{t} \left(e^{-t} - \frac{1}{\lambda^2} \frac{(\mathrm{sh}(\lambda t/2)^2}{(\mathrm{sh}(t/2))^2} \right), \qquad \lambda^2 = \frac{\alpha^2}{4\pi^2}. \tag{21}$$

For the correlator $G_\alpha(x)$ at $\alpha = \pi$ one should use the conjecture (2). The function $f(x) = (1, -1, 1)$ (18) allows for two different representations in the form (1) with two equal exponents $\sim 1/x^{1/2}$: 1) $b_1 = 1/2$, $b_2 = -1/2$, $f_0(x) = e^{i\pi/2}$ and $b_1 = -1/2$, $b_2 = 1/2$, $f_0(x) = e^{-i\pi/2}$, with $x_1 = -\pi/2$, $x_2 = \pi/2$. Thus for the asymptotics one should take the sum of the two terms which gives the following result:

$$G_\pi(x) = \left(e^{ip_F x} + e^{-ip_F x} \right) G_0(x) = \cos(p_F x) 2 G_0(x), \tag{22}$$

where $p_F = \pi/2$ and

$$G_0(x) = \frac{1}{x^{1/2}} \frac{1}{\sqrt{2}} (g(1/2))^2.$$

The correlator $G_\pi(x)$ equal to zero for x - odd and to $\pm 2G_0(x)$ for x- even in accordance with the general properties of the average $G_\pi(x) = \langle \prod_{i=1}^{x} (1 - 2n_i) \rangle$ (real function). The same result for G_π can be obtained using the reduction of the determinant to the square of the half-size determinant. At $\alpha \neq \pi$ the first term $e^{ip_F x}$ in eq.(22) corresponds to the expression (20) while

the second term gives the subleading asymptotics. One can also obtain the asymptotics of the correlator (22) at the arbitrary $p_F \neq \pi/2$:

$$G_\pi(x) = (e^{ip_F x} + e^{-ip_F x}) \frac{1}{x^{1/2}} \frac{1}{\sqrt{\sin(p_F)}} \frac{1}{\sqrt{2}} (g(1/2))^2, \tag{23}$$

which also have two different exponents and at arbitrary p_F is the real function of x. Note that the same results could be obtained from equation (4) by taking the sum of the two terms corresponding to the functions $(1, e^{i\pi}, 1)$, and $(1, e^{-i\pi}, 1)$.

The spin-spin correlator (10) using the functions (19) for the two terms we obtain from eq.(4) the asymptotics

$$G(x) = \frac{1}{x^{1/2}} \frac{1}{\sqrt{2}} (g(1/2))^2, \tag{24}$$

where the constant $g(1/2) = \sqrt{\pi}(G(1/2))^2$, can be represented in the form:

$$\ln(g(1/2)) = \frac{1}{4} \int_0^\infty \frac{dt}{t} \left(e^{-4t} - \frac{1}{(\mathrm{ch}(t))^2} \right).$$

Note that for the functions (19) there is only one representation (1), $b_1 = b_2 = 1/2$ and $b_1 = b_2 = -1/2$ respectively so the conjecture (2) leads to the same results.

Finally, one can use equation (11) and the functions (19) and the discontinuities at the points $x_{1,2} = \pm p_F$ to obtain the asymptotics of the correlator $G(x)$ for the XX spin chain in the magnetic field in the form:

$$G(x) = \frac{1}{x^{1/2}} \frac{\sqrt{\sin(p_F)}}{\sqrt{2}} (g(1/2))^2, \tag{25}$$

which differs from the well known asymptotics (24) by the simple factor $(\sin(p_F))^{1/2} = (1 - h^2)^{1/4}$. Note that both eq.(2) and (4) predict the real positive function $G(x)$ without the oscillating factors in agreement with the observation that the ground-state wave function of the Hamiltonian (6) at $\gamma = 0$ is the positive function of the coordinates.

4. Correlators for the XY spin chain.

Let us start with the calculations of the asymptotics of the xx-correlator $G^{xx}(x) = \langle S_x^x S_0^x \rangle$ for the different values of the parameters γ, h ($\lambda_{1,2}$). In Section 2 this correlator was represented by the determinant corresponding to the function $f_1(x)$ (15). At $h < 1$ we have for the parameters $\lambda_{1,2}$ the condition $|\lambda_{1,2}| < 1$, and this function is the smooth function with zero winding number, so the strong Szego theorem can be applied. Thus in the whole region $h < 1$ the correlator has the long-range order which can be easily calculated:

$$G^{xx}(x) = \frac{1}{4}(1 - a^2)^{1/2}(1 - h^2)^{1/4}, \qquad a = \frac{1-\gamma}{1+\gamma}, \qquad h < 1.$$

Next, let us consider the correlator at the line $h = 1$. At this value of h the parameter $\lambda_2 = 1$ and the function $f_1(x)$ takes the form:

$$f_1(z) = \left(\frac{1 - a/z}{1 - az}\right)^{1/2} \left(\frac{1 - 1/z}{1 - z}\right)^{1/2}, \qquad z = e^{ix}.$$

One can see that this function has the form (1) with the function $f_0(x)$ given by the first factor in the last equation and the single jump discontinuity at $z = 1$ with the parameter $b = -1/2$. The calculations according to the formulas (2), (3) yield the result

$$G^{xx}(x) = \frac{1}{x^{1/4}} \frac{\gamma^{3/4}}{2(1+\gamma)} \sqrt{\pi} (G(1/2))^2, \qquad h = 1,$$

in agreement with the result of ref.[3]. Let us turn to the case $h > 1$. In this case $\lambda_2 > 1$ and the function takes the following form:

$$f_1(z) = z^{-1} \left(\frac{1 - \lambda_1/z}{1 - \lambda_1 z}\right)^{1/2} \left(\frac{1 - \lambda z}{1 - \lambda/z}\right)^{1/2}, \qquad \lambda = 1/\lambda_2 < 1. \tag{26}$$

To calculate the determinant we use the method proposed in ref.[5], which allows one to use the general Fisher-Hartwig conjecture (2), (3). Representing the matrix elements as an integrals in the complex plane over the unit circle, we deform the contour of integration to the circle $|z| = 1/\lambda$ which is the *outer* boundary of the annulus $\lambda < |z| < 1/\lambda$. This procedure is correct since the function (26) is an analytic function inside this annulus and reduces to the substitution $z \to z/\lambda$ in eq.(26). As a result we obtain the function of the form (1):

$$f_0(z) = \lambda \left(\frac{1 - \lambda_1 \lambda/z}{1 - (\lambda_1/\lambda) z}\right)^{1/2} \frac{1}{(1 - \lambda^2/z)^{1/2}}, \qquad a = \frac{1}{4}, \quad b = -\frac{3}{4}$$

with the singularity at the point $z = 1$. Applying the conjecture (2) we obtain the result for the case $h > 1$:

$$G^{xx}(x) = \frac{1}{x^{1/2}} \left(\frac{1}{\lambda_2}\right)^x \frac{1}{4\sqrt{\pi}} (1 - \lambda_1^2)^{1/4} (1 - \lambda_2^{-2})^{-1/4} (1 - \lambda_1 \lambda_2)^{1/2}, \qquad h > 1,$$

in agreement with the corresponding formula in ref.[3].

Let us proceed with the calculations of the yy-correlator $G^{yy}(x) = \langle S_x^y S_0^y \rangle$. The behaviour of this correlator in the four regions in the space of parameters is completely different. We begin with the basic region in the space of parameters where both γ and h are sufficiently small: $h^2 + \gamma^2 < 1$. In this case the parameters $\lambda_{1,2}$ are inside the unit circle at the complex plane:

$$\lambda_{1,2} = (1+\gamma)^{-1} \left(h \pm i\sqrt{1 - h^2 - \gamma^2}\right) = e^{\pm i\phi} \sqrt{a}.$$

In order to obtain an appropriate representation (1) we deform the contour of integration from the unit circle to the circle of the radius $\sqrt{a}$, which amounts to the substitution $z \to \sqrt{a} z$ in the

function $f_2(z)$ (15). One can see that the resulting function $f(z)$ has the two singularities at the points $z = e^{\pm i\phi}$. One can prove that the only possible way to represent this function in the form (1) is as follows. Namely, we single out the two equivalent Fisher-Hartwig singularities at the points $x = \pm\phi$. Thus we have the representation:

$$f_0(z) = a(1 - e^{i\phi}az)^{-1/2}(1 - e^{-i\phi}az)^{-1/2}, \qquad a_1 = a_2 = \frac{1}{4}, \qquad b_1 = b_2 = \frac{3}{4}.$$

Substituting this function into the equations (2), (3) we obtain the correlator:

$$G^{yy}(x) = a^x \frac{1}{x}\left(\frac{1}{2\pi}\sin(\phi)(1-a)^{-1/2}(1 + a^2 - 2a\cos(2\phi))^{-1/4}\right).$$

Next, let us consider the case $h^2 + \gamma^2 > 1$, but $h < 1$, when the parameters λ_1 and λ_2 are real and both are inside the unit circle, $\lambda_1 < \lambda_2 < 1$. Taking the contour to be the inner boundary of the region $\lambda_2 < |z| < 1/\lambda_2$, where the function $f_2(z)$ (15) is an analytic function, we obtain the representation:

$$f_0(z) = \lambda_2^2\left(\frac{1 - (\lambda_1/\lambda_2)/z}{1 - (\lambda_1\lambda_2)z}\right)^{1/2} \frac{1}{(1 - \lambda_2^2 z)^{1/2}}, \qquad a = \frac{1}{4}, \qquad b = \frac{7}{4},$$

which gives the following correlation function:

$$G^{yy}(x) = -\frac{1}{x^3}\left(\lambda_2^2\right)^x \frac{1}{8\pi}(1-\lambda_1^2)^{1/4}(1-\lambda_2^2)^{-3/4}(1-\lambda_1\lambda_2)^{-1/2}(1-\lambda_1/\lambda_2)^{-1}, \qquad h < 1.$$

This expression is identical with that given in ref.[3] which can be easily established using the following useful relations: $1-\lambda_1\lambda_2 = 1-a = 2\gamma/(1+\gamma)$ and $(1-\lambda_1^2)(1-\lambda_2^2) = 4(1-h^2)/(1+\gamma)^2$.

As in the case of the xx-correlator, at the line $h = 1$ no special deformation of the integration contour is required. The parameters $\lambda_{1,2}$ are $\lambda_1 = a$, $\lambda_2 = 1$ and from the function $f_2(x)$ (15) one can easily obtain the representation:

$$f_0(z) = \left(\frac{1 - a/z}{1 - az}\right)^{1/2}, \qquad a = 0, \qquad b = \frac{3}{2},$$

Calculating the constant (3) we get:

$$G^{yy}(x) = -\frac{1}{x^{9/4}}\frac{\sqrt{\pi}}{32}\gamma^{-5/4}(1+\gamma)(G(1/2))^2.$$

Finally, let us consider the region $h > 1$ ($\lambda_2 > 1$). In this case the calculations are analogous to the corresponding case for the xx-correlator. Introducing the parameter $\lambda = 1/\lambda_2 < 1$, we observe that the function is an analytic function in the annulus $\lambda < |z| < 1/\lambda$. Deformation of the integration contour to the *inner* boundary of the annulus, $z \to \lambda z$, gives the representation (1) in the form:

$$f_0(z) = \lambda\left(\frac{1 - (\lambda_1/\lambda)/z}{1 - \lambda_1\lambda z}\right)^{1/2}(1 - \lambda^2 z)^{1/2}, \qquad a = -\frac{1}{4}, \qquad b = \frac{5}{4}.$$

Applying the conjecture (2) we obtain the result:

$$G^{yy}(x) = -\frac{1}{x^{3/2}}\left(\frac{1}{\lambda_2}\right)^x \frac{1}{8\sqrt{\pi}}(1-\lambda_1^2)^{1/4}(1-\lambda_2^{-2})^{3/4}(1-(\lambda_1/\lambda_2))^{-1}(1-\lambda_1\lambda_2)^{-1/2}, \qquad h > 1,$$

in agreement with the corresponding formula in ref.[3]. Let us note that in all the cases considered in this section (except the yy- correlator at $h^2+\gamma^2 < 1$) we have obtained the functions with the single Fisher-Hartwig singularity. In the case of the single singularity the asymptotics (2), (3) was proved in ref.[8]. In conclusion, it is easy to verify that in all the cases considered, there is only one possible way to choose the contour of integration (inner or outer boundary of the annulus). In fact, in all the cases at the opposite boundary the condition $a_r \pm b_r \in Z^-$ is fulfilled. Formally, in this case we have the constant $E = 0$, which suggests, that these special cases should be excluded from the general statement of the conjecture (2), (3) since in this case the matrix elements corresponding to the pure Fisher-Hartwig singularity does not exist.

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